Solutions

Directions:

1. Write your name with one character in each box below.
2. Show all work. No credit for answers without work.
3. You are permitted to use one 8.5 inch by 11 inch sheet of prepared notes. No other aides are allowed.

1. **[15 points]** Given the matrix A and the reduced row echelon form of A, find bases for Col(A) and Nul(A).

$$\begin{array}{cccccccc} \text{col } 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \qquad\qquad \begin{array}{cccccccc} \text{col } 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

$$A = \begin{bmatrix} 0 & -1 & -2 & -5 & 0 & 1 & -1 \\ 0 & -2 & -4 & -10 & -2 & 11 & 14 \\ 0 & 1 & 2 & 5 & 1 & -5 & -6 \\ 0 & 0 & 0 & 0 & 1 & -5 & -9 \end{bmatrix} \cdots \to \cdots \begin{bmatrix} 0 & 1 & 2 & 5 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Col (A): Cols 2, 5, 6: $\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 11 \\ -5 \\ -5 \end{bmatrix}$

Nul (A): Free variables x_1, x_3, x_4, x_7

$$\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{bmatrix} x_1 \\ -2x_3 - 5x_4 - 3x_7 \\ x_3 \\ x_4 \\ -x_7 \\ -2x_7 \\ x_7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix}$$

2. **[10 points]** Let V be the subspace of $\mathbb{R}^4$ containing the vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that $x_1 + x_2 + x_3 + x_4 = 0$ and $x_1 - 2x_2 + 3x_3 - 4x_4 = 0$. Find a basis for V.

Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 3 & -4 \end{bmatrix}$. Want Nul (A).

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 2 & -5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{2}{3} & \frac{5}{3} \end{bmatrix} \rightsquigarrow \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \begin{bmatrix} 1 & 0 & \frac{5}{3} & -\frac{2}{3} \\ 0 & 1 & -\frac{2}{3} & \frac{5}{3} \end{bmatrix}$$

x_3, x_4 free. $x_3 = 3, x_4 = 0$: $\vec{v}_1 = \begin{bmatrix} -5 \\ 2 \\ 3 \\ 0 \end{bmatrix}$

$x_3 = 0, x_4 = 3$ $\vec{v}_2 = \begin{bmatrix} 2 \\ -5 \\ 0 \\ 3 \end{bmatrix}$

$\boxed{\text{So } \begin{bmatrix} -5 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 0 \\ 3 \end{bmatrix} \text{ is a basis for } V.}$

3. **[4 parts, 5 points each]** Compute the determinant of the following matrices.

(a) $\begin{bmatrix} 3 & 1 \\ -2 & -5 \end{bmatrix}$

$(3)(-5) - (1)(-2) = -15 + 2$

$= \boxed{-13}$

(b) $\begin{array}{c} \begin{matrix} + & - & + & - \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & a \\ b & 0 & c & d \\ e & f & g & h \\ i & 0 & j & k \end{bmatrix} \end{array}$

$-a \begin{vmatrix} b & 0 & c \\ e & f & g \\ i & 0 & j \end{vmatrix} = (-a)(-f) \begin{vmatrix} b & c \\ i & j \end{vmatrix}$

$= \boxed{af(bj - ic)}$

(c) $\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & -1 \\ 0 & 5 & -1 \end{bmatrix}$

$1 \begin{vmatrix} 1 & -1 \\ 5 & -1 \end{vmatrix} - 2 \begin{vmatrix} 3 & -1 \\ 5 & -1 \end{vmatrix} + 0 \begin{vmatrix} \cdots \end{vmatrix}$

$= (1)(-1) - (-1)(5) - 2[-3 + 5]$

$= -1 + 5 - 4 = \boxed{0}$

(d) $\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} = A$

let $c = \det(A)$. We use row reduction keeping track of the determinant at each step:

$$\underset{-c}{\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}} \rightsquigarrow \underset{-c}{\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}} \rightsquigarrow \underset{-c}{\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & -1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}}$$

$$\rightsquigarrow \underset{-c}{\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -4 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}} \rightsquigarrow \underset{-c}{\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -4 & -1 \end{bmatrix}} \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

So $-c = -5 \Rightarrow \det(A) = c = \boxed{5}$

4. **[5 points]** Let A be an $n \times n$ matrix such that $A^2 = I_n$. Prove that $\det(A)$ equals 1 or -1.

We know $\det(AB) = \det(A)\det(B)$. So $1 = \det(I_n) = \det(A^2) = \det(A) \cdot \det(A)$.

Since $(\det(A))^2 - 1 = 0$, we have $(\det(A) + 1)(\det(A) - 1) = 0$, and this implies

$\det(A) \in \{-1, 1\}$. ∎

5. **[15 points]** Diagonalize the following matrix if possible by finding an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. There is no need to compute P^{-1}. If diagonalization is not possible, then explain why.

$$A = \begin{bmatrix} -5 & 8 & 0 \\ -4 & 7 & 0 \\ 8 & -16 & -1 \end{bmatrix}$$

Find eigenvalues:

$$|A - \lambda I| = \begin{vmatrix} -5-\lambda & 8 & 0 \\ -4 & 7-\lambda & 0 \\ 8 & -16 & -1-\lambda \end{vmatrix} = (-1-\lambda)\begin{vmatrix} -5-\lambda & 8 \\ -4 & 7-\lambda \end{vmatrix} = -(\lambda+1)\left[(-5-\lambda)(7-\lambda)+32\right] = -(\lambda+1)\left[\lambda^2-2\lambda-3\right]$$

$$= -(\lambda+1)\left[(\lambda-3)(\lambda+1)\right] = -(\lambda+1)^2(\lambda-3).$$ So $\lambda=-1$, mult 2 and $\lambda=3$.

$\underline{\lambda=-1}$: $\begin{bmatrix} -4 & 8 & 0 \\ -4 & 8 & 0 \\ 8 & -16 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$\underline{\lambda=3}$: $\begin{bmatrix} -8 & 8 & 0 \\ -4 & 4 & 0 \\ 8 & -16 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 2 & -4 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad v_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

Let $P = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$

$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$

6. **[15 points]** Give a formula for A^k, where $A = \begin{bmatrix} -4 & 3 \\ -6 & 5 \end{bmatrix}$.

$(-4-\lambda)(5-\lambda) - (-6)(3) = 0$

$\lambda^2 - \lambda - 20 + 18 = 0$

$\lambda^2 - \lambda - 2 = 0$

$(\lambda-2)(\lambda+1) = 0$

$\underline{\lambda_1=-1}$: $\begin{bmatrix} -3 & 3 \\ -6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{\lambda_2=2}$: $\begin{bmatrix} -6 & 3 \\ -6 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Let $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \; D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

and note $P^{-1} = \frac{1}{2-1}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

We have $A^k = (PDP^{-1})^k = PD^kP^{-1}$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} (-1)^k & 0 \\ 0 & 2^k \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} (-1)^k \cdot 2 & (-1)^{k+1} \\ -2^k & 2^k \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 2(-1)^k - 2^k & (-1)^{k+1} + 2^k \\ 2(-1)^k - 2^{k+1} & (-1)^{k+1} + 2^{k+1} \end{bmatrix}}$$

7. **[5 parts, 3 points each]** True/False. Justify your answer.

   (a) If A and B are $n \times n$ matrices, then $\det(A + B) = \det(A) + \det(B)$.

   $\boxed{\text{FALSE}}$. If $A = B = I_n$, then $\det(A+B) = \det(2I_n) = 2^n$ but $\det(A) + \det(B) = 2\det(I_n) = 2$.

   (b) Matrices A and B are similar if and only if A and B have the same eigenvalues with the same multiplicities.

   $\boxed{\text{FALSE}}$. The matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ have the same characteristic polynomial $(\lambda-1)^2$ but are not similar, as I_2 is only similar to itself.

   (c) Let A be a matrix with 7 rows and 10 columns. If the null space of A has dimension 4, then the column space of A has dimension 6.

   $\boxed{\text{True}}$. The rank theorem implies $\text{rank}(A) + \dim(\text{Nul}(A)) = 10$. Since $\dim(\text{Nul}(A)) = 4$, we have $\text{rank}(A) = 10 - 4 = 6$. Since $\text{rank}(A) = \dim(\text{Col}(A))$, the claim is true.

   (d) If A is a square matrix and r is a scalar, then A is similar to rA.

   $\boxed{\text{FALSE}}$. The matrix I_n is not similar to $2I_n$.

   (e) If A and B are similar matrices, then A^{-1} and B^{-1} are also similar.

   $\boxed{\text{True}}$. If $A = PBP^{-1}$, then $A^{-1} = (PBP^{-1})^{-1} = (P^{-1})^{-1} B^{-1} P^{-1} = PB^{-1}P^{-1}$, and so A^{-1} and B^{-1} are similar.

8. **[5 points]** Find a matrix B such that $B^2 = \underbrace{\begin{bmatrix} -1 & 5 \\ -10 & 14 \end{bmatrix}}_{A}$.

Diagonalize A: $(-1-\lambda)(14-\lambda) - (-10)(5) = \lambda^2 - 13\lambda - 14 + 50 = \lambda^2 - 13\lambda + 36 = (\lambda - 9)(\lambda - 4)$.

$\lambda_1 = 9$: $\begin{bmatrix} -10 & 5 \\ -10 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$ $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda_2 = 4$: $\begin{bmatrix} -5 & 5 \\ -10 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So $A = \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}}_{D} \underbrace{\left(\frac{1}{1-2}\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}\right)}_{P^{-1}}$

$= \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$

Take $B = PD^{1/2}P^{-1}$ so that $B^2 = PD^{1/2}P^{-1}PD^{1/2}P^{-1} = PDP^{-1} = A$.

So $B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 3 \\ 4 & -2 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}}$.

Check: $\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ -10 & 14 \end{bmatrix}$ ✓