Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[2 parts, 2 points each]** Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and let $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   (a) Apply the power method to compute $\mathbf{x}_k$ and μ_k for $0 \le k \le 3$.

   $Ax_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mu_0 = 1$, $x_1 = \frac{1}{\mu_0} Ax_0 = 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   $Ax_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mu_1 = 1$, $x_2 = \frac{1}{\mu_1} Ax_1 = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   $Ax_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mu_2 = 1$, $x_3 = \frac{1}{\mu_2} Ax_2 = 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   $Ax_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mu_3 = 1$

   (b) Note that $\mathbf{x}_k$ is not approaching the direction of an eigenvector of A. Why does this not contradict the power method?

   The power method requires that A have a strictly dominant eigenvalue. The eigenvalues of A are 1 and −1 (since A is triangular) and these have the same magnitude. Therefore A lacks a strictly dominant eigenvalue.

2. **[4 points]** Given $\mathbf{y}$ and $\mathbf{v}$ below, decompose $\mathbf{y}$ as $\mathbf{y} = c\mathbf{v} + \mathbf{z}$ where c is a scalar and $\mathbf{z} \cdot \mathbf{v} = 0$.

$$\mathbf{y} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$$

$y \cdot v = c\, v \cdot v$ so

$$c = \frac{y \cdot v}{v \cdot v} = \frac{(2)(-3) + (2)(1) + (1)(-2)}{9 + 1 + 4} = \frac{-6}{14} = \frac{-3}{7}.$$

$$z = y - cv = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \left(\frac{-3}{7}\right) \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} = \frac{1}{7}\left(\begin{bmatrix} 14 \\ 14 \\ 7 \end{bmatrix} + \begin{bmatrix} -9 \\ 3 \\ -6 \end{bmatrix} \right) = \frac{1}{7} \begin{bmatrix} 5 \\ 17 \\ 1 \end{bmatrix}.$$

3. **[2 points]** Let $W = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Prove that if $\mathbf{z} \cdot \mathbf{v}_i = 0$ for $1 \leq i \leq p$, then $z \in W^{\perp}$.

Let $w \in W$. We have that $w = c_1 v_1 + \cdots + c_p v_p$ for some scalars $c_1, \ldots, c_p$.

Now $z \cdot w = z \cdot (c_1 v_1 + \cdots + c_p v_p) = c_1 (z \cdot v_1) + c_2 (z \cdot v_2) + \cdots + c_p (z \cdot v_p)$

$= c_1 0 + c_2 0 + \cdots + c_p 0$

$= 0$.

Since $z \cdot w = 0$, z is orthogonal to w. Since z is orthogonal to each $w \in W$, we have $z \in W^{\perp}$.